

Bose-Einstein condensation of a Knudsen gas

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Abstract

We reconcile a long-standing controversy regarding the transition temperature of the Bose-Einstein condensation in a dilute interacting Bose gas, by showing that there is a crossover between ideal gas and interacting gas. The former corresponds to a Knudsen (or collisionless) regime, in which the mean-free-path is much larger than the system dimension, while the latter corresponds to the opposite hydrodynamic regime. The deviation of the transition temperature from that of the non-interacting gas is proportional to \sqrt{a} in the Knudsen regime, and a in the hydrodynamic regime, where a is the scattering length. This crossover may be observable in a Bose gas trapped in a potential or on an optical lattice.

The transition temperature T_c of a dilute interacting Bose gas in 3D has been a subject of much theoretical interest and controversy. Let us denote its fractional deviation from that of the non-interacting gas by

$$\Delta \equiv \frac{T_c - T_c^{(0)}}{T_c^{(0)}}, \quad (1)$$

where $T_c^{(0)} = \frac{2\pi\hbar^2}{mk_B} [n\zeta(3/2)]^{-2/3}$, $\zeta(3/2) \approx 2.612$, $n = N/L^3$, with N the number of particles and L the linear size of the system. Using an argument based on the virial expansion in the high-temperature phase, the author [1] obtained the following result:

$$\begin{aligned} \Delta_0 &= c_0 \sqrt{an^{1/3}}, \\ c_0 &= \frac{8\sqrt{2\pi}}{3[\zeta(3/2)]^{2/3}} \approx 3.527. \end{aligned} \quad (2)$$

where a is the S-wave scattering length. This has been rederived in a mean-field approach [2]. Earlier, Toyada [3] got the same result but with an opposite sign, by summing one-loop graphs. Many other calculations, which approach

the transition point from the low-temperature side, obtain a different behavior, with a linear dependence on a [4]–[8]:

$$\Delta_1 = c_1 a n^{1/3}. \quad (3)$$

The constant c_1 is of order 1.

The purpose of this note is to point out that there is a crossover between ideal gas and interacting gas. The former corresponds to a Knudsen (or collisionless) regime, in which the mean-free-path in the condensate is much larger than the size of the system, while the latter corresponds to the opposite hydrodynamic regime. The fractions Δ_0 and Δ_1 apply in the Knudsen and hydrodynamic regime, respectively.

Eq.(2) for Δ_0 results from a straightforward generalization of the method used in the ideal gas, namely, by calculating the maximum number of particles the gas phase can accommodate. That is, one locates the onset of condensation in the gas phase. The value (3) for Δ_1 , on the other hand, is obtained by approaching the transition from below, within the condensed phase. These approaches lead to different answers because of a peculiar property of the perturbation theory of Bose-condensed systems, namely, the expansion parameter crosses over from a at extremely low densities to a fractional power of a at finite densities.

The ground-state energy per particle of a uniform dilute Bose gas in conventional perturbation theory is, to second order in a , given by [9] [10]

$$\frac{E_0}{(\hbar^2/2m)N} = 4\pi a n \left(1 + A_2 \frac{a}{L}\right) \quad (4)$$

where [11] [12]

$$A_2 = 2.837297 \dots. \quad (5)$$

In the same expansion scheme, the excitation spectrum has an energy gap $8\pi a n \xi$, where ξ is the condensate fraction, and there are no phonons. This expansion breaks down in the thermodynamic limit $N \rightarrow \infty$, $L \rightarrow \infty$, with $n = N/L^3$ held fixed. While the correction term a/L vanishes, one finds that higher-order corrections diverge.

A systematic graphical analysis [10] reveals that the dimensionless expansion parameter is

$$\frac{aN}{L} \sim \sqrt{\frac{\text{System size}}{\text{Mean-free-path in condensate}}}. \quad (6)$$

The mean-free-path in the condensate is of order $(n\sigma)^{-1}$, with scattering cross section $\sigma \sim Na^2$, where the factor N comes from Bose enhancement. The perturbation series has the general structure

$$\begin{aligned} \frac{E_0}{(\hbar^2/2m)N} = 4\pi an &+ \frac{1}{NL^2} \left[A_2 \left(\frac{aN}{L} \right)^2 + A_3 \left(\frac{aN}{L} \right)^3 + \dots \right. \\ &+ \frac{B_2}{N} \left(\frac{aN}{L} \right)^2 + \frac{B_3}{N} \left(\frac{aN}{L} \right)^3 + \dots \\ &\left. + \frac{C_2}{N^2} \left(\frac{aN}{L} \right)^2 + \frac{C_3}{N^2} \left(\frac{aN}{L} \right)^3 + \dots \right] \end{aligned} \quad (7)$$

This expansion obviously fails in the thermodynamic limit, where aN/L diverges. To obtain a finite energy per particle, one reorganizes the perturbation series by effectively performing infinite sums in each horizontal line in (7). One can see on general grounds that the first horizontal sum should be proportional to $a^{5/2}$. Explicit calculation gives an expansion in powers of $\sqrt{na^3}$ in the thermodynamic limit [13]:

$$\frac{E_0}{(\hbar^2/2m)N} \longrightarrow 4\pi an \left[1 + \frac{128}{15\sqrt{\pi}} \sqrt{na^3} + O(na^3) \right]. \quad (8)$$

The excited spectrum now consists of phonons, with no energy gap. The expansion scheme (7), which is valid for the N -body system with fixed N in an infinite volume, yields the virial series for the equation of state [14], and that leads to Δ_0 given by (2). The partial summation of the perturbation series leading to (8) is equivalent to the Bogoliubov transformation, with the emergence of “pairing”, and long-wavelength collective modes that contribute significantly to Δ_1 in (3).

The Knudsen regime is relevant for kinetic processes, such as effusion through a hole, when the mean-free-path is much larger than the dimension L of the hole. For a macroscopic system in thermal equilibrium, L is the size of the whole system, and the Knudsen region consists of an infinitesimally small neighborhood of zero density, and is of little experimental interest.

For a Bose gas condensed in a potential or on an optical lattice, the Knudsen regime spans a region accessible to experimentation, for one can vary the system size, particle number, and even the scattering length through Feshbach resonances. In a harmonic trap of frequency ω_0 , the relevant dimension is the harmonic length

$$L = \sqrt{\frac{\hbar}{m\omega_0}}. \quad (9)$$

The trapped gas behaves like an ideal gas when $aN/L \ll 1$, and crosses over to the Thomas-Fermi regime when $aN/L \gg 1$. During the crossover, the size of the condensate increases from the ideal-gas radius

$$R_0 \sim L \quad (10)$$

to the Thomas-Fermi radius [15]

$$R_{\text{TF}} \sim \left(\frac{aN}{L} \right)^{1/5} L. \quad (11)$$

The excitations change from ideal-gas states in the trap to collective oscillations, the lowest ones being the “breathing” (monopole) and quadrupole modes [16]. Note that the ideal gas regime is not the same as that of no interaction. Even a very weak interaction leads to quantum phase coherence that is absent in the non-interacting system.

As illustration, take $a = 1$ nm, $L = 100$ μ m. The critical particle number is

$$N_c = \frac{L}{a} = 10^5. \quad (12)$$

The transition temperature depends on N , and interpolates between two fractional deviations:

$$\Delta(N) = \begin{cases} \Delta_0 & (N \ll N_c) \\ \Delta_1 & (N \gg N_c) \end{cases}. \quad (13)$$

The values given in (2) and (3) are for the uniform gas, but may be used for order-of-magnitude estimates. The former corresponds to a 1% correction, while the latter amount to 0.02%. The Knudsen gas has a higher transition temperature, because it is harder to excite the condensate in that case, owing to the energy gap.

References

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